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# SEQUENTIAL VOTING WHEN LONG ELECTIONS ARE COSTLY

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This paper analyzes sequential voting in binary elections when voters are motivated by a desire both to elect their preferred candidate and to avoid a long and costly election. I find a unique equilibrium in which a voter's action depends both on the intensity of the voter's preferences as well as how well the candidates have done in earlier voting rounds. This equilibrium results in momentum in which voters are more likely to vote for the candidate currently in the lead. Furthermore, the probability a voter votes for a candidate is increasing in the size of the candidate's lead. As a consequence, a candidate is more likely to win the election if the candidate's stronger supporters vote earlier in the election.

## 1. INTRODUCTION

SEQUENTIAL VOTING games are games in which some voters decide how to vote after observing how other voters have voted. Such games are an important aspect of many electoral processes. For example, presidential primaries in the United States are typically decided by a sequential voting process in which voters in some states vote well before other states have their elections. It is widely thought that voters in later voting rounds may be influenced by the results of early voting rounds and may be more likely to vote for candidates that have done well early in the election.[1]

This paper introduces a novel modeling assumption that gives rise to the momentum effects described in the empirical literature. Specifically, I assume that voters are motivated both by a desire to avoid a long election as well as a desire to elect their preferred candidate. In the essentially unique equilibrium, a voter conditions his or her vote on both the intensity of his or her preferences as well as how well the candidates have done in earlier voting rounds. I show that some voters have an incentive to vote against their private preferences by voting for whichever candidate is in the lead. Moreover, the fraction of voters that will choose to vote for the candidate currently in the lead is increasing in the size of the candidate's lead. Candidates who do well early in the election thus obtain momentum and increase their chances of doing well later in the election. As a result of this momentum, I show that a candidate is more likely to win an election if it happens to be the case that the candidate's stronger supporters vote earlier in the election.

Why would voters care about avoiding a long election? There is ample empirical evidence that a long and divisive party primary ultimately hurts the chances that the nominee from that party will go on to win the general election (Buell, 1986; Kenney and Rice, 1987; Lengle, 1980; Lengle et al., 1995; Southwell, 1986; Stone, 1984, 1986).[2] The

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[1] Empirical examples of this phenomenon can be found in Bartels (1985, 1988), Kenney and Rice (1994), Knight and Schiff (2007), and Popkin (1994).

[2] While this is conventional wisdom, it is worth noting that not all studies have drawn this conclusion. Stone et al. (1992) present evidence that divisive party primaries may draw more primary voters into the process and make more people available for the party during the general election. Miller et al. (1988) also question whether divisive party primaries hurt a party, but their study is in the context of gubernatorial primaries. By contrast, my paper focuses on elections with sequential voting that are seen in presidential primaries.

underlying theme in this literature is that if a party has a lengthy primary, then candidates in that primary will spend an extended amount of time attacking each other in their battle for their party's nomination, while the nominee from the opposing party is able to focus on campaigning for the general election. This tends to hurt the party with a lengthy primary in several ways.

For one, the party that becomes embroiled in a lengthy primary tends to lose ground in general election polls during this time (Kenney and Rice, 1987; Lengle et al., 1995). For example, the 1976 Republican presidential primary proved to be an extremely long and hard-fought contest, and the identity of the Republican nominee was not known until the start of the Republican convention. As a result of this, the Republican nominee, Gerald Ford, began the 1976 general election campaign with a monumental deficit in the polls, and Ford's aides felt that this made it extremely difficult for a Republican to win the general election (Kenney and Rice, 1987). Similarly, Democratic party elites believed that Democratic candidate Walter Mondale was hurt by the relatively lengthy Democratic primary in the 1984 U.S. presidential election (Kenney and Rice, 1987).

Another problem is that candidates who must fight a lengthy primary battle typically find it harder to secure the votes of voters in their party in the general election (Kenney and Rice, 1987; Lengle, 1980; Southwell, 1986; Stone, 1986). For example, Democratic candidate Jimmy Carter felt that he had to spend far more time securing the support of Democratic voters in the 1980 U.S. presidential election than he would have had he not faced such a difficult challenge in the Democratic primary (Carter, 1982).

A final related reason a lengthy and divisive party primary might hurt the nominee from that party is that it makes it more difficult to obtain the assistance of party activists whose support is typically needed for a successful general election campaign. For example, Stone (1984, 1986) gives evidence from the 1980 U.S. presidential election that indicates that a divisive primary meant that activists who supported the losing candidate from the primary were relatively less likely to actively help their party's nominee get elected. Also, Buell (1986) finds evidence of this phenomenon for the 1984 Democratic presidential primary.[3] All of these factors can lead a party to prefer that its primary end as quickly as possible.

The concern that a lengthy primary hurts a candidate's chances of winning an election is not limited to academics and people actively involved in politics. For example, polls taken during the 2008 U.S. presidential election revealed that a majority of Democratic voters believed that the lengthy Democratic primary would ultimately hurt their party's chances of winning the 2008 election, and these voters would have preferred their party's primary to end more quickly.[4] Thus avoiding a lengthy primary is a real concern that voters have explicitly stated that they faced in a recent U.S. presidential election.

I know of only two other papers that discuss circumstances that give rise to the momentum effects described in the applied literature. Ali and Kartik (2008) present a theory in which voters choose to vote for a candidate that has done well early in the election because a candidate's strong early performance gives evidence that this candidate is more likely to be the better candidate. Also, Callander (2007) illustrates that if voters derive utility from voting for the winning candidate, then there is an equilibrium in which voters ultimately vote for the candidate that is winning.

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[3] Buell (1986) also acknowledges that the 1984 Democratic presidential primary brought some new activists into the process and that some of these activists supported the Democratic candidate in the general election.

[4] This was noted in an April 2008 MSNBC poll.

While these papers are both interesting, there are significant differences between my paper and the above papers beyond my novel assumption that long elections are costly. My paper is one of pure private values, whereas the above papers are concerned with common values. I also find a unique equilibrium, whereas the above papers focus on a particular equilibrium that gives rise to momentum.

More importantly, the dynamics of momentum are different in my model and the above models. In Ali and Kartik (2008) and Callander (2007), a candidate who obtains momentum keeps the momentum for the remainder of the election, as voters vote uninformatively for the remainder of the election by voting for whichever candidate has the momentum. By contrast, in my model a candidate who has the momentum is more likely to do well in future elections, but the opposing candidate may still regain the momentum with a series of strong performances after the initial setbacks. My model thus allows for richer dynamics involving shifts in momentum from one candidate to the other over time.

This distinction is significant because there have been several presidential primaries in which momentum has passed from one candidate to the other over time. For example, in the 1980 Republican presidential primary, Bush was thought to have the momentum after his initial victory in Iowa, but Reagan was thought to regain the momentum after strong performances in the next few states. In the 1984 Democratic presidential primary, Hart obtained the momentum after early successes in New Hampshire and Vermont, but Mondale ultimately regained the momentum in later states.[5] More recently, in the 2008 Democratic presidential primary, Obama was thought to have the momentum after his early win in Iowa, Clinton was thought to regain the momentum after her successes in New Hampshire and Nevada, and Obama was thought to regain the momentum once again following a string of successes after Super Tuesday.

Other papers on sequential voting focus on different issues than why voters would choose to vote for candidates that have done well in early voting rounds in binary elections. Dekel and Piccione (2000) have shown that, under standard modeling assumptions in binary elections, informative equilibria under simultaneous voting are also equilibria under sequential voting. Sequential voting, therefore, need not cause voters to adopt different strategies than they would employ under simultaneous voting. Battaglini (2005) and Battaglini et al. (2007) consider elections with two candidates in which voting is costly and voters are allowed to abstain, but they are more concerned with the relative efficiency and equity of simultaneous and sequential voting. Hummel (2009) and Morton and Williams (1999, 2001) consider the dynamics of elections with three candidates. Finally, Aldrich (1980), Klumpp and Polborn (2006), and Strumpf (2002) consider how sequential voting affects campaign expenditures in later voting rounds, but do not explicitly consider voter strategies.[6]

## 2. THE MODEL

There are two candidates, $A$ and $B$, and a set of voters $N = \{1, \ldots, n\}$, where $n$ is odd. The voters vote one after another. I assume that voter 1 votes without observing how any other voters have voted, and that each voter $i > 1$ votes after observing how voters $1, \ldots, i-1$ have voted.

The game ends when some candidate first achieves a total of $(n+1)/2$ votes. If candidate $X$ first achieves a total of $(n+1)/2$ votes after voter $j$'s vote, then voter $i$'s payoff

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[5] One can find extended discussions of these primaries in Bartels (1988) and Popkin (1994).

[6] It is worth noting that Klumpp and Polborn (2006) also present a model in which momentum arises endogenously and momentum can shift from one candidate to the other.

for the game is $u_i(X) - jc$, where $u_i(X)$ is the voter's utility from electing candidate $X$, and $c \geq 0$ is some exogenously given cost. This cost reflects the fact that voters would like the election to end as quickly as possible.[7]

Throughout the paper I let $v_i \equiv u_i(A) - u_i(B)$ denote the difference between a voter's utility from electing candidate $A$ and a voter's utility from electing candidate $B$. I also normalize $u_i(B)$ to $u_i(B) = 0$ and assume that each $v_i$ is an independent random variable drawn from an atomless cumulative distribution function $F_i$ satisfying $F_i(-x) = 1 - F_i(x)$ for all $x$. This last assumption means that it is equally likely that a voter prefers $A$ to $B$ by some given amount as it is that the voter prefers $B$ to $A$ by the same amount. Finally, I let $Z_i$ denote the support of $F_i$.

Each voter knows his or her utility from electing candidates $A$ or $B$ but does not know the utilities of any of the other voters. Voter $i$ may be called on to vote at any history of votes for players $1, \ldots, i-1$ in which both $A$ and $B$ have not already received $(n+1)/2$ votes. If $H_i$ denotes the set of all such histories, then a strategy for player $i$ is a mapping $\sigma_i$: $H_i \times Z_i \to [0, 1]$, where $\sigma_i$ is the probability with which a voter casts a vote for $A$. I characterize equilibrium strategies in the next section.

## 3. PRELIMINARIES

Throughout the remainder of the paper it will be useful to define the tally $m$ at a given decision node to be the difference between the number of players that have already voted for $A$ and the number of players that have already voted for $B$. A player's optimal action at a given point in the game will depend crucially on the running tally when the player is called on to vote.

In deciding how to vote, a player must consider how his or her vote affects two things which may be relevant to the player's payoff. The first thing that a player must consider is that a vote may affect the probability that the various candidates are elected. The second thing a player must consider is that a vote may affect the expected length of the election. In deciding how to vote, a player compares the relative costs and benefits from these two types of incentives.

I now state the first result of the paper:

**Proposition 1.** (a) There is an essentially unique equilibrium characterized by a set of cutpoints $v_i^*(m)$.[8] In this equilibrium, voter $i$ votes for $A$ when the tally is $m$ if $v_i \geq v_i^*(m)$ and votes for $B$ otherwise.

(b) The probability that $A$ wins the election when voter $i$ is called on to vote, $q_i$, and the expected number of additional players that vote in the game when voter $i$ is called on to vote, $k_i$, can both be written as functions of only the tally $m$.

(c) The probability that $A$ wins the election when voter $i$ is called on to vote at a tally $m$, $q_i(m)$, is non-decreasing in $m$.

Part (a) of this proposition guarantees that when a player decides how to vote, the player only needs condition on the number of the previous players that have voted for each of the two candidates, and not the entire history that was used to reach this tally. This result

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[7] If $c < 0$, then voters would prefer that the election be as long as possible. In this case, one would obtain comparative statics results opposite of those in my paper.

[8] By essentially unique, I mean that the equilibrium is unique up to the action taken by a voter with ideal point $v_i^*(m)$, who is indifferent between voting for $A$ and $B$ in equilibrium.

is logical since a player's payoff is only affected by the length of the election and the identity of the candidate that is elected, rather than the precise history used to reach that electoral outcome.

Similarly, part (b) guarantees that the probability $A$ wins the election after a certain history, as well as the expected number of additional players that vote in the game after a certain history, are each only functions of the tally at that history, and not the entire history used to reach that tally. Part (c) reflects the intuitive fact that the probability $A$ wins the election cannot decrease as a result of $A$ having earned more votes. I let $q_i(m)$ denote the probability that $A$ wins the election when voter $i$ is called on to vote at a tally of $m$, and let $k_i(m)$ denote the expected number of additional players that vote in the game when voter $i$ is called on to vote at a tally of $m$.

In both Proposition 1 and the remainder of the manuscript, $v_i^*(m)$ is defined to be the unique value of $v_i$ such that player $i$ is indifferent between voting for $A$ and voting for $B$. In particular, if player $i+1$ may be called on to vote at both tallies of $m+1$ and $m-1$, then $v_i^*(m) = (k_{i+1}(m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1))$. Similar expressions for $v_i^*(m)$ arise when player $i+1$ will not have to vote at both of these tallies because, for instance, a vote for $A$ by player $i$ at a tally of $m$ elects $A$ and ends the election.

If $v_i^*(m) = 0$, then player $i$ votes for $A$ if and only if player $i$ prefers $A$ to $B$. But if $v_i^*(m) \neq 0$, then players may have an incentive to vote against their private preferences. In particular, if $v_i^*(m) > 0$, then a player who prefers $A$ to $B$ might vote for $B$, and if $v_i^*(m) < 0$, then a player who prefers $B$ to $A$ might vote for $A$. The further $v_i^*(m)$ is from zero, the more likely it is that a player may have an incentive to vote against his or her private preferences.

The fact that player $i$ is more likely to act according to his or her private preferences when $v_i^*(m)$ is closer to zero makes the fact that $v_i^*(m) = (k_{i+1}(m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1))$ intuitive. $k_{i+1}(m+1) - k_{i+1}(m-1)$ represents the difference between the expected length of the election after player $i$ votes for $A$ and the expected length of the election after player $i$ votes for $B$. If this difference decreases in magnitude, there is a relatively weaker incentive to try to influence the length of the election, and the players are relatively more willing to vote according to their private preferences. This is reflected by the fact that if $k_{i+1}(m+1) - k_{i+1}(m-1)$ decreases in magnitude, $v_i^*(m) = (k_{i+1}(m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1))$ becomes closer to zero, and players thus employ strategies that are closer to the ones they would employ if there were no cost to long elections.

Similarly, $q_{i+1}(m+1) - q_{i+1}(m-1)$ represents the difference between the probability $A$ wins the election after player $i$ votes for $A$ and the probability $A$ wins the election after player $i$ votes for $B$. If this difference increases in magnitude, there is a relatively stronger incentive to try to influence the outcome of the election, and the players are relatively more willing to vote according to their private preferences. This is reflected by the fact that if $q_{i+1}(m+1) - q_{i+1}(m-1)$ increases in magnitude, $v_i^*(m) = (k_{i+1}(m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1))$ becomes closer to zero, and players thus employ strategies that are closer to the ones they would employ if there were no cost to long elections. Thus the fact that $v_i^*(m) = (k_{i+1}(m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1))$ reflects an intuitive tradeoff between trying to influence the length of the election and trying to influence the outcome of the election.

It should be noted that, in principle, it is possible that $v_i^*(m) = \infty$ or $v_i^*(m) = -\infty$. In the first of these cases, player $i$ votes for $B$ with probability 1, and in the second of these cases, player $i$ votes for $A$ with probability 1. Such situations arise when a vote has no effect on the probability the various candidates are elected, but does affect the expected length of the election.

Now that the logic of the equilibrium has been established, I give my second preliminary result:

**Proposition 2.** (a) The expected number of additional players that vote in the game when voter $i$ is called on to vote at a tally of $m$, $k_i(m)$, satisfies $k_i(m) = k_i(-m)$.

(b) The probability that $A$ wins the election when voter $i$ is called on to vote at a tally $m$, $q_i(m)$, satisfies $q_i(m) + q_i(-m) = 1$.

(c) $k_i(m)$ is non-increasing in $|m|$.

Parts (a) and (b) of this proposition indicate that the expected length of the game and the probability a given candidate is elected have a natural symmetry with respect to the current tally. If a given candidate trails by a certain amount, then the expected length of the election is the same as it would be if the other candidate trailed by the same amount. Furthermore, if candidate $A$ trails by a certain amount, then the probability $A$ wins is the same as the probability $B$ would win if $B$ trailed by the same amount.

These results both follow from the fact that $F_i(-x) = 1 - F_i(x)$ for all $i$ and $x$. Because of this, it is equally likely that a voter prefers $A$ to $B$ by some given amount as it is that the voter prefers $B$ to $A$ by the same amount, and there is a natural symmetry to the properties of the game.

Part (c) of this proposition indicates that as the size of a candidate's lead over the other candidate increases, the expected length of the election decreases. Intuitively, if the size of a candidate's lead over the other candidate increases, then the candidate needs fewer additional votes to win the election, the election is closer to being over, and the expected length of the election is smaller. Thus $k_i(m)$ is non-increasing in $|m|$.

## 4. MAIN RESULTS

This section gives comparative statics results that show how the properties of the game are affected by the assumption that long elections are costly. In particular, I show that voters may have an incentive to vote against their private preferences by voting for candidates that have outperformed their opponent early in the election. I further show that voters have a stronger incentive to vote for candidates currently in the lead if the size of a candidate's lead increases. I conclude by illustrating that the probability a candidate is elected is greater if the candidate is fortunate to have the candidate's strongest supporters vote earlier in the election.

I first give a result on the values of $v_i^*(m)$:

**Proposition 3.** $v_i^*(-m) = -v_i^*(m)$ for all $m$, $v_i^*(m) \geq 0$ for $m \leq 0$, and $v_i^*(m) \leq 0$ for $m \geq 0$.

The fact that $v_i^*(-m) = -v_i^*(m)$ for all $m$ indicates that a voter's incentives to vote for a given candidate are symmetric with respect to tally. If a voter who prefers $A$ to $B$ by a certain amount has an incentive to vote for $A$ when $A$ is leading by a given amount, then a voter who prefers $B$ to $A$ by the same amount has an incentive to vote for $B$ when $B$ is leading by the same amount. As in parts (a) and (b) of Proposition 2, this symmetry follows from the fact that $F_i(-x) = 1 - F_i(x)$ for all $i$ and $x$. This result also implies that $v_i^*(0) = 0$ for all $i$.

The fact that $v_i^*(m) \geq 0$ for $m \leq 0$, and $v_i^*(m) \leq 0$ for $m \geq 0$ indicates that if a voter votes against his or her private preferences, then the voter will do so by voting for the candidate currently in the lead. This result makes sense intuitively. If a voter wishes to vote against his or her private preferences, then the voter does so because the voter believes the election will end more quickly if he or she votes against his or her private preferences. The best way for a voter to try to make the election end more quickly is by casting a vote for the candidate that is more likely to win. Since the candidate currently in the lead is more likely to win the election, if a voter wishes to vote against his or her private preferences, the voter will vote for whichever candidate is currently in the lead.

It should be noted that the inequalities in this proposition will normally be strict. In particular, as long as $c \neq 0$, then $v_i^*(m) \neq 0$ for $m \neq 0$. Thus the voters who do not have particularly intense preferences will typically have an incentive to vote against their private preferences by voting for the leading candidate. This result thus indicates that candidates who have done well early in the election can obtain momentum by getting a higher percentage of votes from future voters than they would expect to get otherwise.

While a candidate who is currently in the lead will have the momentum, this momentum is not a guarantee of success in the future, and the opposing candidate may regain momentum by doing well in future contests and regaining the lead. Thus while momentum improves a candidate's chances of success, it does not preclude the possibility that the candidate will lose the momentum in the future.

I now illustrate how the importance of this momentum changes with the size of a candidate's lead:

**Proposition 4.** $v_i^*(m)$ is non-increasing in $m$.

The fact that $v_i^*(m)$ is non-increasing in $m$ means that the probability voters vote for $A$ is weakly increasing in the number of votes that $A$ has received so far. Thus if $A$ is in the lead, voters have a stronger incentive to vote for $A$ if the size of $A$'s lead increases. Similarly, if $B$ is in the lead, then voters have a lesser incentive to vote for $B$ if the size of $B$'s lead decreases. Understanding why this result is true therefore amounts to understanding why voters have a stronger incentive to vote for a candidate if the size of the candidate's lead increases.

To understand the intuition behind this, suppose that $A$ has a very large lead when voter $i$ is called on to vote. In that case, $A$ is almost certain to win the election regardless of whether the voter votes for $A$ or not. Thus a voter has relatively little effect on the probability that $A$ wins the election. A voter who votes for $B$ will thus mostly only serve to delay $A$'s inevitable victory. Given that voting for $B$ would mostly only serve to unnecessarily lengthen the election, voters will have a strong incentive to vote for $A$. Thus in this case, only the voters who intensely prefer $B$ to $A$ will wish to vote for $B$, and the cutoff $v_i^*(m)$ will be very negative.

Now suppose that $A$ only has a small lead when voter $i$ is called on to vote. In this case, there is still a fair amount of uncertainty about who will win the election, and a single vote is likely to have a relatively large effect on the probabilities with which the various candidates win the election. At the same time, there is a substantial chance that not voting for $A$ would actually serve to shorten the election. Since there is still a significant chance that $B$ will win the election, voter $i$ may be able to shorten the election by voting for $B$ in the circumstances in which $B$ would ultimately go on to win the election. Thus voting for $B$ will not increase the expected length of the election as much as it did when $A$ had a very large lead. Thus attempting to influence the outcome of the election is relatively more important when the election is still close than it is when one candidate has a large lead, and voters are more likely to vote according to their private preferences when the election is closer. For this reason, the cutoff $v_i^*(m)$ will be much closer to zero when $m$ is closer to zero.

With this reasoning in mind, it is intuitive that players will have a stronger incentive to vote for the leading candidate the further that candidate is in the lead. Thus $v_i^*(m)$ is non-increasing in $m$.

I now illustrate how a player's vote affects the strategic incentives of the voter that votes immediately afterwards:

**Proposition 5.** If player $i$ may be called on to vote at a tally of $m$ and player $i+1$ may be called on to vote at a tally of $m+1$, then $v_i^*(m) \geq v_{i+1}^*(m+1)$. If player $i$ may be called on to vote at a tally of $m$ and player $i+1$ may be called on to vote at a tally of $m-1$, then $v_i^*(m) \leq v_{i+1}^*(m-1)$.

This result indicates that if player $i$ votes for $A$, then player $i+1$ will have at least as strong an incentive to vote for $A$ as player $i$ did. Similarly, if player $i$ votes for $B$, then player $i+1$ will have a weaker incentive to vote for $A$ than player $i$ did. This result thus indicates that momentum shifts in predictable ways in response to the results of a previous vote. A voter responds to a candidate's success in the prior round of voting by becoming more willing to vote for that candidate than the voter who voted in the previous round.

The intuition behind this result is similar to the intuition behind Proposition 4. If $A$ is in the lead, then an extra vote for $A$ by player $i$ increases $A$'s lead, and gives future voters a stronger incentive to vote for $A$ by the same reasoning as in Proposition 4. Similarly, if $B$ is in the lead, and player $i$ votes for $A$, then the size of $B$'s lead decreases, and future voters will be more willing to act according to their private preferences. In either case, player $i+1$ will have a stronger incentive to vote for $A$ than player $i$ did if player $i$ votes for $A$. Similar reasoning indicates that player $i+1$ will have a weaker incentive to vote for $A$ than player $i$ did if player $i$ votes for $B$.

Using this result, I show how a candidate's probability of victory is affected by placing a candidate's stronger supporters earlier in the election:

**Proposition 6.** Suppose $v > v'$. Then the probability that $A$ wins the election if $v_i = v$ and $v_{i+1} = v'$ is at least as high as the probability that $A$ wins the election if $v_i = v'$ and $v_{i+1} = v$.

Proposition 6 indicates that $A$ is at least as likely to win the election if a voter who likes $A$ better comes just before a voter who does not like $A$ as much as if the voter who does not like $A$ as much comes just before the voter who likes $A$ better. Similarly, any finite number of these switches can only increase $A$'s chances of winning the election. Thus if a certain voter likes $A$ relatively better, the candidate benefits from having this voter vote earlier in the election.

This result makes sense given the momentum effects I have described in the previous propositions. If a voter who likes $A$ relatively better votes earlier in the election, and this voter votes for $A$, then this will increase the chances that the voter who did not like $A$ as much will vote for $A$. However, if the voter who likes $A$ relatively better had voted later, then this voter would not have been able to increase the chances that the other voter would vote for $A$. Thus it is to a candidate's advantage if the earlier draws of voter preferences are stronger supporters than the later draws.

## 5. CONCLUSION

This paper has analyzed a model of sequential voting in which voters wish to both elect their preferred candidate and avoid a long and costly election. The results indicate that voters that do not have particularly intense preferences over candidates have an incentive to vote for whichever candidate is currently in the lead so as to decrease the expected length of the election. The model thus gives rise to momentum effects in which candidates who have done well early in the election can expect to obtain more votes later in the election. As a result of this, a candidate is more likely to win the election if the candidate's stronger supporters vote earlier in the election than if these supporters vote later.

While success in early voting rounds improves a candidate's chances of doing well in later voting rounds, success in early voting rounds does not guarantee victory in later voting rounds. The opposing candidate may still regain momentum by winning a series of contests and parlay this momentum into an ultimate victory. I now discuss the robustness of the results to changes in modeling assumptions.

In the present model voting is completely sequential in the sense that voters are able to observe how all previous voters have voted when they cast their vote. While this is unrealistic in many elections, the results can be immediately extended to an analogus model in which there are aspects of both simultaneous and sequential voting.

Suppose, for example, that there are $n$ voting rounds and $r$ voters in each round, where $r$ is odd. Also suppose that a candidate wins a given round of voting if at least $(r+1)/2$ voters vote for the candidate in that voting round, a candidate is elected if the candidate wins $(n+1)/2$ rounds of voting, and a voter's payoff if candidate $X$ wins the election in voting round $j$ is $u(X)-jc$, where $u(X)$ denotes the voter's utility from electing candidate $X$. Finally, suppose that each voter's utility difference between the two candidates is an independent random variable such that, in the $i$th voting round, the value of the median voter's utility difference is an independent random variable drawn from the distribution $F_i$.

In this case, the equilibrium construction is identical to that in the current paper. In particular, it is an equilibrium for a voter in voting round $i$ to vote for $A$ if and only if the voter's utility difference between $A$ and $B$ is at least $v_i^*(m)$, where $m$ denotes the difference between the number of rounds of voting $A$ has won so far and the number of rounds of voting $B$ has won so far. If the voters follow these strategies, the probability a candidate wins a given voting round at a given tally is the same as that in the model considered in the paper, and all of the comparative statics results regarding momentum hold in this framework as well. Thus, the theory in the present paper has relevance for elections that have aspects of both simultaneous and sequential voting.

Another possible extension of the current model regards what happens when voters who vote late in the election do not mind a lengthy election as much. For example, in the context of presidential primaries, voters who vote in the late primary states may be enthused by the rare event in which the election remains undecided when they vote and they have a chance to actually cast a deciding vote. However, the equilibrium in this paper also remains an equilibrium in a corresponding model in which voters who vote late in the election gain utility from being able to cast a vote that may make a difference in the election.

Suppose, for example, that a voter in the $i$th voting round obtains an additional utility of $\pi_i \geq 0$ if the voter has a chance to cast a vote that might make a difference. Thus if candidate $X$ wins the election in voting round $j < i$, then voter $i$ obtains a utility of $u_i(X) - jc$, and if candidate $X$ wins the election in voting round $j \geq i$, then voter $i$ obtains a utility of $u_i(X) + \pi_i - jc$. In this case, the unique equilibrium is identical to the equilibrium characterized in the present paper. In fact, regardless of the utility a voter who votes in the $i$th round obtains from the result of an election that ends before the $i$th voting round, as long as the voter's utility decreases linearly with the length of the election conditional on the election lasting at least $i$ rounds, then the strategies in this paper remain an equilibrium. Thus the results in this paper are robust to a model in which voters also care about whether their vote might make a difference.

Finally, I wish to discuss how the results would extend to a model in which voters care about avoiding a divisive election, rather than a lengthy election. Much of the applied literature on the fact that a lengthy party primary tends to hurt the party's chances of winning the general election also discusses the fact that a divisive party primary can hurt the party's chances of winning the general election, where divisiveness is defined by the closeness of the election (see for example Kenney and Rice (1987); Lengle (1980); Lengle et al. (1995)). Thus it is interesting to ask if one can obtain similar results when voters care about the candidate's margin of victory in the election.

Again the results are robust to a corresponding model that makes this assumption. Let $z$ denote a candidate's margin of victory, where margin of victory is defined as the difference between the number of votes the winner received and the number of votes the loser received. Also suppose that if candidate $X$ wins the election with margin of victory $z$, then voter $i$ obtains a utility of $u_i(X) + zc$, so that a voter's utility is increasing in margin of victory and decreasing in the closeness of the election. Note that if a candidate wins the election in the $j$th voting round, then $z = n + 1 - j$. Thus if candidate $X$ wins the election in the $j$th voting round, then voter $i$ obtains a utility of $u_i(X) - jc + (n + 1)c$. Since this differs from the utility in the present paper only by a constant, the unique equilibrium in this game is the same as the unique equilibrium in my model. The results in this paper are thus robust to a variety of natural changes to the model.

I now discuss one possibility for future research. In the model in this paper, voters only consider how a shortened election affects their own utility rather than also considering how a shorter election benefits other voters. Thus voters may take actions that, on average, lead to a longer primary than is optimal for the primary voters as a whole. Given this, party leaders who are concerned about maximizing the wellbeing of the voters in their primary may have an incentive to try to take actions that end the primary sooner than it would end normally. For example, these considerations might affect when superdelegates optimally declare allegiance to a candidate. Further research could address how these incentives affect the actions of party leaders during presidential primaries.

## APPENDIX

*Proof of Proposition 1.* The proof is by induction on $i$. First consider the base case $i = n$. Note that player $n$ is only called on to vote when exactly $(n-1)/2$ voters have voted for $A$, $(n-1)/2$ voters have voted for $B$, and the tally is $m = 0$. In this case, player $n$'s payoff from voting for $A$ is $u_n(A) - nc$ and player $n$'s payoff from voting for $B$ is $u_n(B) - nc$. Thus player $n$ prefers to vote for $A$ if and only if $u_n(A) - nc \geq u_n(B) - nc$ or $u_n(A) - u_n(B) \geq 0$ or $v_n \geq 0$. Since this strategy is independent of the precise history used to reach the tally $m = 0$, part (a) holds for voter $n$.

Now note that if player $n$ is called on to vote at the tally $m = 0$, then the expected number of additional players that will vote in the game is exactly 1. Also note that the probability $A$ wins the election is the probability that $v_n \geq 0$ or $\frac{1}{2}$. Since this probability and this expected number of players are both independent of the precise history used to reach the tally $m = 0$, part (b) holds for voter $n$.

Finally note that part (c) holds vacuously for voter $n$ because voter $i$ is only ever called on to vote at one particular tally. Thus the results in parts (a)–(c) all hold for voter $n$.

Now suppose that the results in parts (a)–(c) all hold for voter $i + 1$ and use this to prove that the results in parts (a)–(c) hold for voter $i$. I let $q_{i+1}(m)$ denote the probability that $A$ will win the election when voter $i + 1$ is called on to vote at a tally of $m$ and let $k_{i+1}(m)$ denote the expected number of additional players that will vote in the game when voter $i + 1$ is called on to vote a tally of $m$. I first use the inductive hypothesis to prove that parts (a) and (b) hold for voter $i$. I consider three cases:

*Case 1*: Suppose that the tally is $m = n - i$ when voter $i$ is called on to vote. In this case, $(n - 1)/2$ of the first $i - 1$ voters have voted for $A$ and $i - (n + 1)/2$ of the first $i - 1$ voters have voted for $B$. Thus if voter $i$ votes for $A$, then $A$ wins the election, and no additional voters after voter $i$ are forced to vote. If voter $i$ votes for $B$, then voter $i + 1$ will be called on to vote at a tally of $n - i - 1$, $A$ wins with probability $q_{i+1}(n - i - 1)$, and the expected number of additional players that will vote in the game after $i$ has voted is $k_{i+1}(n - i - 1)$.

Thus player $i$'s expected payoff from voting for $A$ is $u_i(A) - ic$ and player $i$'s expected payoff from voting for $B$ is $q_{i+1}(n - i - 1)u_i(A) + (1 - q_{i+1}(n - i - 1))u_i(B) - (i + k_{i+1}(n - i - 1))c$. From this it follows that player $i$ prefers to vote for $A$ if and only if $u_i(A) - ic \geq q_{i+1}(n - i - 1)u_i(A) + (1 - q_{i+1}(n - i - 1))u_i(B) - (i + k_{i+1}(n - i - 1))c$ or $(1 - q_{i+1}(n - i - 1))(u_i(A) - u_i(B)) \geq -k_{i+1}(n - i - 1)c$ or $v_i \geq -k_{i+1}(n - i - 1)c/(1 - q_{i+1}(n - i - 1))$. Since this strategy is independent of the precise history used to reach the tally $m = n - i$, part (a) holds for voter $i$ in this case.

Since player $i$ votes for $A$ if and only if $v_i \geq -k_{i+1}(n - i - 1)c/(1 - q_{i+1}(n - i - 1))$, player $i$ votes for $A$ with probability $1 - F_i(-k_{i+1}(n - i - 1)c/(1 - q_{i+1}(n - i - 1)))$. Thus $A$ wins the election with probability $1 - F_i(-k_{i+1}(n - i - 1)c/(1 - q_{i+1}(n - i - 1))) + F_i(-k_{i+1}(n - i - 1)c/(1 - q_{i+1}(n - i - 1)))q_{i+1}(n - i - 1)$ when voter $i$ is called on to vote at a tally of $n - i$. Also, the expected number of additional players that vote in the game when voter $i$ is called on to vote at a tally of $n - i$ is $1 + F_i(-k_{i+1}(n - i - 1)c/(1 - q_{i+1}(n - i - 1)))k_{i+1}(n - i - 1)$. Since this probability and this expected number of players are both independent of the precise history used to reach the tally $m = n - i$, part (b) holds for voter $i$ in this case.

*Case 2*: Suppose that the tally is $i - n$ when voter $i$ is called on to vote. In this case $(n - 1)/2$ of the first $i - 1$ voters have voted for $B$ and $i - (n + 1)/2$ of the first $i - 1$ voters have voted for $A$. Thus if voter $i$ votes for $B$, then $B$ wins the election, and no additional voters after voter $i$ are forced to vote. If voter $i$ votes for $A$, then voter $i + 1$ will be called on to vote at a tally of $i - n + 1$, $A$ wins with probability $q_{i+1}(i - n + 1)$, and the expected number of additional players that will vote in the game after $i$ has voted is $k_{i+1}(i - n + 1)$.

Thus player $i$'s expected payoff from voting for $A$ is $q_{i+1}(i - n + 1)u_i(A) + (1 - q_{i+1}(i - n + 1))u_i(B) - (i + k_{i+1}(i - n + 1))c$ and player $i$'s expected payoff from voting for $B$ is $u_i(B) - ic$. From this it follows that player $i$ prefers to vote for $A$ if and only if $q_{i+1}(i-n+1)u_i(A)+(1-q_{i+1}(i-n+1))u_i(B)-(i+k_{i+1}(i-n+1))c \geq u_i(B)-ic$ or $q_{i+1}(i-n+1)(u_i(A)-u_i(B)) \geq k_{i+1}(i-n+1)c$ or $v_i \geq k_{i+1}(i-n+1)c/q_{i+1}(i-n+1)$. Since this strategy is independent of the precise history used to reach the tally $m=i-n$, part (a) holds for voter $i$ in this case.

Since player $i$ votes for $A$ if and only if $v_i \geq k_{i+1}(i-n+1)c/q_{i+1}(i-n+1)$, player $i$ votes for $A$ with probability $1-F_i(k_{i+1}(i-n+1)c/q_{i+1}(i-n+1))$. Thus $A$ wins the election with probability $(1-F_i(k_{i+1}(i-n+1)c/q_{i+1}(i-n+1)))q_{i+1}(i-n+1)$ when voter $i$ is called on to vote at a tally of $i-n$. Also, the expected number of additional players that vote in the game when voter $i$ is called on to vote at a tally of $i-n$ is $1+(1-F_i(k_{i+1}(i-n+1)c/q_{i+1}(i-n+1)))k_{i+1}(i-n+1)$. Since this probability and this expected number of players are both independent of the precise history used to reach the tally $m=i-n$, part (b) holds for voter $i$ in this case.

*Case 3*: Suppose that voter $i$ is called on to vote at some tally $m$ satisfying $i-n<m<n-i$. If voter $i$ votes for $A$, then voter $i+1$ will be called on to vote at a tally of $m+1$, $A$ wins with probability $q_{i+1}(m+1)$, and the expected number of additional players that will vote in the game after $i$ has voted is $k_{i+1}(m+1)$. If voter $i$ votes for $B$, then voter $i+1$ will be called on to vote at a tally of $m-1$, $A$ wins with probability $q_{i+1}(m+1)$, and the expected number of additional players that will vote in the game after $i$ has voted is $k_{i+1}(m+1)$.

Thus player $i$'s expected payoff from voting for $A$ is $q_{i+1}(m+1)u_i(A)+(1-q_{i+1}(m+1))u_i(B)-(i+k_{i+1}(m+1))c$, and player $i$'s expected payoff from voting for $B$ is $q_{i+1}(m-1)u_i(A)+(1-q_{i+1}(m-1))u_i(B)-(i+k_{i+1}(m-1))c$. From this it follows that player $i$ prefers to vote for $A$ if and only if $q_{i+1}(m+1)u_i(A)+(1-q_{i+1}(m+1))u_i(B)-(i+k_{i+1}(m+1))c \geq q_{i+1}(m-1)u_i(A)+(1-q_{i+1}(m-1))u_i(B)-(i+k_{i+1}(m-1))c$ or $(q_{i+1}(m+1)-q_{i+1}(m-1))(u_i(A)-u_i(B)) \geq (k_{i+1}(m+1)-k_{i+1}(m-1))c$ or $v_i \geq (k_{i+1}(m+1)-k_{i+1}(m-1))c/(q_{i+1}(m+1)-q_{i+1}(m-1))$, where the last inequality follows from the inductive hypothesis that $q_{i+1}(m+1) \geq q_{i+1}(m-1)$. Since this strategy is only a function of the tally $m$, part (a) holds for voter $i$ in this case.

Since player $i$ votes for $A$ if and only if $v_i \geq (k_{i+1}(m+1)-k_{i+1}(m-1))c/(q_{i+1}(m+1)-q_{i+1}(m-1))$, it follows that player $i$ votes for $A$ with probability $1-F_i((k_{i+1}(m+1)-k_{i+1}(m-1))c/(q_{i+1}(m+1)-q_{i+1}(m-1)))$. Thus $A$ wins the election with probability $(1-F_i((k_{i+1}(m+1)-k_{i+1}(m-1))c/(q_{i+1}(m+1)-q_{i+1}(m-1)))$ $q_{i+1}(m+1)+F_i((k_{i+1}(m+1)-k_{i+1}(m-1))c/(q_{i+1}(m+1)-q_{i+1}(m-1)))q_{i+1}(m-1)$ when voter $i$ is called on to vote at a tally of $m$. Also, the expected number of additional players that vote in the game when voter $i$ is called on to vote at a tally of $m$ is $1+(1-F_i((k_{i+1}(m+1)-k_{i+1}(m-1))c/(q_{i+1}(m+1)-q_{i+1}(m-1)))k_{i+1}(m+1)+F_i((k_{i+1}(m+1)-k_{i+1}(m-1))c/(q_{i+1}(m+1)-q_{i+1}(m-1)))k_{i+1}(m-1)$. Thus the result in part (b) holds for voter $i$ in this case. From this it follows that the results in parts (a) and (b) hold for voter $i$.

Now I prove that the result in part (c) holds for voter $i$. Note that if player $i$ votes for $A$ at some tally $m$, then $A$ wins with probability $q_{i+1}(m+1)$ if $m<n-i$ and wins with probability 1 if $m=n-i$. Also, if player $i$ votes for $B$ at some tally $m$, then $A$ wins with probability $q_{i+1}(m-1)$ if $m>i-n$ and wins with probability 0 if $m=i-n$. Since $0 \leq q_{i+1}(m-1) \leq q_{i+1}(m+1) \leq 1$, it follows that $A$ is at least as likely to win the election if player $i$ votes for $A$ as if player $i$ votes for $B$.

Now consider some tally $m$ such that player $i$ may be called on to vote at both the tallies $m$ and $m-2$. From the above paragraph, we know that the probability that $A$ wins the election when player $i$ is called on to vote at the tally $m$ is equal to at least the probability that $A$ wins the election if player $i$ votes for $B$ at the tally $m$ or $q_{i+1}(m-1)$. Also, the probability that $A$ wins the election when player $i$ is called on to vote at the tally $m-2$ is no greater than the probability that $A$ wins the election when player $i$ votes for $A$ at the tally $m-2$ or $q_{i+1}(m-1)$. Thus $q_i(m) \geq q_i(m-2)$ indeed holds and the result in part (c) holds for voter $i$. By induction, it follows that all the results in parts (a), (b), and (c) hold. ■

*Proof of Proposition 2*. The proof is by induction on $i$. First consider the base case $i=n$. Note that parts (a) and (c) hold vacuously because player $i=n$ is only ever called on to vote at a tally of $m=0$. Now it was noted in the proof of Proposition 1 that the probability $A$ wins the election when voter $n$ is called on to vote at a tally of $m=0$ is $\frac{1}{2}$. Thus $q_n(0)=\frac{1}{2}$, and $q_n(m)+q_n(-m)=1$ holds for all tallies $m$ at which player $n$ may be called on to vote. Thus part (b) holds for player $n$ as well, and the results all hold for player $n$.

Now suppose the results holds for voter $i+1$ and use this to prove the results hold for voter $i$. Recall from the proof of Proposition 1 that $k_i(n-i) = 1 + F_i(-k_{i+1}(n-i-1)c/(1-q_{i+1}(n-i-1)))k_{i+1}(n-i-1)$ and $k_i(i-n) = 1 + (1-F_i(k_{i+1}(i-n+1)c/(q_{i+1}(i-n+1))))k_{i+1}(i-n+1)$. By the induction hypothesis, $k_{i+1}(i-n+1)=k_{i+1}(n-i-1)$ and $q_{i+1}(i-n+1)=1-q_{i+1}(n-i-1)$. Thus $k_i(i-n) = 1 + (1-F_i(k_{i+1}(n-i-1)c/(1-q_{i+1}(n-i-1))))k_{i+1}(n-i-1) = 1 + F_i(-k_{i+1}(n-i-1)c/(1-q_{i+1}(n-i-1)))k_{i+1}(n-i-1)$, where the last equality follows from the fact that $F_i(-x)=1-F_i(x)$ for all $i$ and $x$. But this means that $k_i(i-n)=k_i(n-i)$. Thus the result in part (a) holds for $m=i-n$ for player $i$.

Also recall from the proof of Proposition 1 that

$$\begin{aligned} k_i(m) = 1 &+ \left(1 - F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)\right)k_{i+1}(m+1) \\ &+ F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)k_{i+1}(m-1) \end{aligned}$$

if $i-n < m < n-i$. Thus

$$\begin{aligned} k_i(-m) = 1 &+ \left(1 - F_i\left(\frac{(k_{i+1}(-m+1)-k_{i+1}(-m-1))c}{q_{i+1}(-m+1)-q_{i+1}(-m-1)}\right)\right)k_{i+1}(-m+1) \\ &+ F_i\left(\frac{(k_{i+1}(-m+1)-k_{i+1}(-m-1))c}{q_{i+1}(-m+1)-q_{i+1}(-m-1)}\right)k_{i+1}(-m-1). \end{aligned}$$

By the induction hypothesis, $k_{i+1}(-m+1)=k_{i+1}(m-1)$, $k_{i+1}(-m-1)=k_{i+1}(m+1)$, $q_{i+1}(-m+1)=1-q_{i+1}(m-1)$, and $q_{i+1}(-m-1)=1-q_{i+1}(m+1)$. Thus

$$\begin{aligned} k_i(-m) = 1 &+ \left(1 - F_i\left(\frac{(k_{i+1}(m-1)-k_{i+1}(m+1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)\right)k_{i+1}(m-1) \\ &+ F_i\left(\frac{(k_{i+1}(m-1)-k_{i+1}(m+1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)k_{i+1}(m+1) \\ = 1 &+ \left(1 - F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)\right)k_{i+1}(m+1) \\ &+ F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)k_{i+1}(m-1), \end{aligned}$$

where the last equality follows from the fact that $F_i(-x)=1-F_i(x)$ for all $i$ and $x$. But this means that $k_i(m)=k_i(-m)$ if $i-n<m<n-i$. Thus the result in part (a) holds for player $i$.

Now I prove that part (b) holds for player $i$. Recall from the proof of Proposition 1 that

$$q_i(n-i)=1-F_i\left(\frac{-k_{i+1}(n-i-1)c}{1-q_{i+1}(n-i-1)}\right)+F_i\left(\frac{-k_{i+1}(n-i-1)c}{1-q_{i+1}(n-i-1)}\right)q_{i+1}(n-i-1)$$

and

$$q_i(i-n)=\left(1-F_i\left(\frac{k_{i+1}(i-n+1)c}{q_{i+1}(i-n+1)}\right)\right)q_{i+1}(i-n+1).$$

By the induction hypothesis, $k_{i+1}(i-n+1)=k_{i+1}(n-i-1)$ and $q_{i+1}(i-n+1)=1-q_{i+1}(n-i-1)$. Thus

$$\begin{aligned}q_i(i-n)&=\left(1-F_i\left(\frac{k_{i+1}(n-i-1)c}{1-q_{i+1}(n-i-1)}\right)\right)(1-q_{i+1}(n-i-1))\\&=F_i\left(\frac{-k_{i+1}(n-i-1)c}{1-q_{i+1}(n-i-1)}\right)(1-q_{i+1}(n-i-1)),\end{aligned}$$

where the equality follows from the fact that $F_i(-x)=1-F_i(x)$ for all $i$ and $x$. But this means that $q_i(n-i)+q_i(i-n)=1$. Thus the result in part (b) holds for $m=i-n$ for player $i$.

Also recall from the proof of Proposition 1 that

$$\begin{aligned}q_i(m)=&\left(1-F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)\right)q_{i+1}(m+1)\\&+F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)q_{i+1}(m-1)\end{aligned}$$

if $i-n<m<n-i$. Thus

$$\begin{aligned}q_i(-m)=&\left(1-F_i\left(\frac{(k_{i+1}(-m+1)-k_{i+1}(-m-1))c}{q_{i+1}(-m+1)-q_{i+1}(-m-1)}\right)\right)q_{i+1}(-m+1)\\&+F_i\left(\frac{(k_{i+1}(-m+1)-k_{i+1}(-m-1))c}{q_{i+1}(-m+1)-q_{i+1}(-m-1)}\right)q_{i+1}(-m-1).\end{aligned}$$

By the induction hypothesis, $k_{i+1}(-m+1)=k_{i+1}(m-1)$, $k_{i+1}(-m-1)=k_{i+1}(m+1)$, $q_{i+1}(-m+1)=1-q_{i+1}(m-1)$, and $q_{i+1}(-m-1)=1-q_{i+1}(m+1)$. Thus

$$\begin{aligned}q_i(-m)=&\left(1-F_i\left(\frac{(k_{i+1}(m-1)-k_{i+1}(m+1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)\right)(1-q_{i+1}(m-1))\\&+F_i\left(\frac{(k_{i+1}(m-1)-k_{i+1}(m+1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)(1-q_{i+1}(m+1))\\=&F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)(1-q_{i+1}(m-1))\\&+\left(1-F_i\left(\frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\right)\right)(1-q_{i+1}(m+1)),\end{aligned}$$

where the last equality follows from the fact that $F_i(-x) = 1 - F_i(x)$ for all $i$ and $x$. But this means that $q_i(m) + q_i(-m) = 1$ if $i - n < m < n - i$. Thus the result in part (b) holds for player $i$.

Now I prove that part (c) holds for player $i$. I restrict attention to the case where $m \geq 0$. The case in which $m \leq 0$ then follows immediately from the fact that $k_i(m) = k_i(-m)$. Note that if player $i$ votes for $A$ at some tally $m \geq 0$, then the expected number of additional players that vote after $i$ is $k_{i+1}(m+1)$ if $m < n - i$ and 0 if $m = n - i$. Also, if player $i$ votes for $B$ at some tally $m \geq 0$, then the expected number of additional players that vote after $i$ is $k_{i+1}(m-1)$. By the induction hypothesis, $k_{i+1}(m-1) \geq k_{i+1}(m+1) \geq 0$. Thus the expected number of additional players that vote after $i$ is at least as high if $i$ votes for $B$ as it is if $i$ votes for $A$.

Now consider some tally $m \geq 0$ such that player $i$ may be called on to vote at both the tallies $m$ and $m + 2$. From the above paragraph, we know the expected number of additional players that vote after player $i$ votes at the tally $m$ is equal to at least the expected number of additional players that vote after player $i$ votes for $A$ or $k_{i+1}(m+1)$. Also, the expected number of additional players that vote after player $i$ votes at the tally $m + 2$ is no greater than the expected number of additional players that vote after player $i$ votes for $B$ or $k_{i+1}(m+1)$. Thus $k_i(m) \geq k_i(m+2)$ indeed holds and the result in part (c) holds for voter $i$. By induction, it follows that all the results in parts (a), (b), and (c) hold. ■

*Proof of Proposition 3*. First note that the result holds for player $i = n$. Player $n$ is only ever called on to vote at a tally of $m = 0$, and it was noted in the proof of Proposition 1 that $v_n^*(0) = 0$. From this it follows immediately that $v_n^*(-m) = -v_n^*(m)$ for all $m$, $v_n^*(m) \geq 0$ for $m \leq 0$, and $v_n^*(m) \leq 0$ for $m \geq 0$.

Now I restrict attention to $i < n$. Recall from the proof of Proposition 1 that $v_i^*(n-i) = -k_{i+1}(n-i-1)c/(1 - q_{i+1}(n-i-1))$, $v_i^*(i-n) = k_{i+1}(i-n+1)c/\ q_{i+1}(i-n+1)$, and $v_i^*(m) = (k_{i+1}(m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1))$ if $i - n < m < n - i$. Thus if $i - n < m < n - i$, then

$$v_i^*(-m) = \frac{(k_{i+1}(-m+1) - k_{i+1}(-m-1))c}{q_{i+1}(-m+1) - q_{i+1}(-m-1)} = \frac{(k_{i+1}(m-1) - k_{i+1}(m+1))c}{q_{i+1}(m+1) - q_{i+1}(m-1)},$$

where the second equality follows from the fact that $k_{i+1}(m) = k_{i+1}(-m)$ and $q_{i+1}(m) + q_{i+1}(-m) = 1$ for all $m$. Thus

$$v_i^*(-m) = -\frac{(k_{i+1}(m+1) - k_{i+1}(m-1))c}{q_{i+1}(m+1) - q_{i+1}(m-1)} = -v_i^*(m)$$

if $i - n < m < n - i$. Also note that

$$\begin{aligned} v_i^*(i-n) &= \frac{k_{i+1}(i-n+1)c}{q_{i+1}(i-n+1)} = \frac{k_{i+1}(n-i-1)c}{1 - q_{i+1}(n-i-1)} = -\frac{-k_{i+1}(n-i-1)c}{1 - q_{i+1}(n-i-1)} \\ &= -v_i^*(n-i). \end{aligned}$$

Thus $v_i^*(-m) = -v_i^*(m)$ holds for all $m$.

Now note that $v_i^*(0) = (k_{i+1}(1) - k_{i+1}(-1))c/(q_{i+1}(1) - q_{i+1}(-1)) = 0$ since $k_{i+1}(1) = k_{i+1}(-1)$. Also, if $0 < m < n - i$, then $v_i^*(m) = (k_{i+1}\ (m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1)) \leq 0$ since $q_{i+1}(m+1) \geq q_{i+1}(m-1)$ and $k_{i+1}(m+1) \leq k_{i+1}(m-1)$ for $m > 0$. Finally note that $v_i^*(n-i) = -k_{i+1}(n-i-1)c/(1 - q_{i+1}(n-i-1)) \leq 0$ since $k_{i+1}(n-i-1) \geq 0$. Thus $v_i^*(m) \leq 0$ for $m \geq 0$. The fact that $v_i^*(m) \geq 0$ for $m \leq 0$ then follows from the fact that $v_i^*(-m) = -v_i^*(m)$ for all $m$ and $v_i^*(m) \leq 0$ for $m \geq 0$. ■

**Lemma 1.** Suppose $w \geq 0$, $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x/w \geq z/y$. Then $x/w \geq (x+z)/(w+y) \geq z/y$.

*Proof.* $x/w \geq z/y$ holds if and only if $xy \geq wz$. $x/w \geq (x+z)/(w+y)$ holds if and only if $x(w+y) \geq w(x+z)$, which holds if and only if $xy \geq wz$. Also, $(x+z)/(w+y) \geq z/y$ holds if and only if $(x+z)y \geq z(w+y)$, which also holds if and only if $xy \geq wz$. Thus $x/w \geq z/y$, $x/w \geq (x+z)/(w+y)$, and $(x+z)/(w+y) \geq z/y$ are all equivalent to $xy \geq wz$, and $x/w \geq z/y$ implies $x/w \geq (x+z)/(w+y) \geq z/y$. ■

*Proof of Proposition 4.* The proof is by induction on $i$. I first prove the cases $i=n$, $i=n-1$, and $i=n-2$ separately. The base case $i=n$ is vacuously true since player $n$ is only ever called on to vote at a tally of $m=0$. The case $i=n-1$ holds because player $n-1$ is only ever called on to vote at tallies of $m=1$ or $m=-1$ and we know from Proposition 3 that $v_{n-1}^*(1) \leq 0 \leq v_{n-1}^*(-1)$. Also, the case $i=n-2$ holds because player $n-2$ is only ever called on to vote at tallies of $m=2$, $m=0$, or $m=-2$, and we know from Proposition 3 that $v_{n-1}^*(2) \leq 0 = v_{n-1}^*(0) = 0 \leq v_{n-1}^*(-2)$.

So suppose the result holds for player $i+1$ and use this to show the result must hold for player $i > n-2$. Recall from the proof of Proposition 1 that $v_i^*(m) = k_{i+1}(i-n+1)c/q_{i+1}(i-n+1)$ if $m=i-n$, and $v_i^*(m) = (k_{i+1}(m+1) - k_{i+1}(m-1))c/(q_{i+1}(m+1) - q_{i+1}(m-1))$ if $i-n < m < n-i$. Note that if $p_i(m)$ denotes the probability with which voter $i$ votes for $A$ at a tally of $m$, then $k_i(m) = 1 + p_i(m)k_{i+1}(m+1)$ if $m=i-n$, and $k_i(m) = 1 + p_i(m)k_{i+1}(m+1) + (1-p_i(m))k_{i+1}(m-1)$ if $i-n < m < n-i$. Also note that $q_i(m) = p_i(m)q_{i+1}(m+1)$ if $m=i-n$, and $q_i(m) = p_i(m)q_{i+1}(m+1) + (1-p_i(m))q_{i+1}(m-1)$ if $i-n < m < n-i$.

Thus if $i+1-n < m-1 < m+1 < n-i-1$, then $k_{i+1}(m+1) - k_{i+1}(m-1) = p_{i+1}(m+1)(k_{i+2}(m+2) - k_{i+2}(m)) + (1-p_{i+1}(m-1))(k_{i+2}(m) - k_{i+2}(m-2))$ and $q_{i+1}(m+1) - q_{i+1}(m-1) = p_{i+1}(m+1)(q_{i+2}(m+2) - q_{i+2}(m)) + (1-p_{i+1}(m-1))(q_{i+2}(m) - q_{i+2}(m-2))$. If $i+1-n = m-1 < m+1 < n-i-1$, then $k_{i+1}(m+1) - k_{i+1}(m-1) = p_{i+1}(m+1)(k_{i+2}(m+2) - k_{i+2}(m)) + (1-p_{i+1}(m-1))k_{i+2}(m)$ and $q_{i+1}(m+1) - q_{i+1}(m-1) = p_{i+1}(m+1)(q_{i+2}(m+2) - q_{i+2}(m)) + (1-p_{i+1}(m-1))q_{i+2}(m)$.

Consider a tally $m$ such that $m+2 < 0$ and player $i$ may be called on to vote at the tallies $m$ and $m+2$. I seek to show that $v_i^*(m) \geq v_i^*(m+2)$. There are several cases.

*Case 1*: Suppose $m=i-n$. Then

$$\begin{aligned} v_i^*(m) &= \frac{k_{i+1}(i-n+1)c}{q_{i+1}(i-n+1)} = \frac{(1+p_{i+1}(i-n+1)k_{i+2}(i-n+2))c}{p_{i+1}(i-n+1)q_{i+2}(i-n+2)} \\ &\geq \frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)} \end{aligned}$$

and

$$\begin{aligned} v_i^*(m+2) &= \frac{(k_{i+1}(i-n+3) - k_{i+1}(i-n+1))c}{q_{i+1}(i-n+3) - q_{i+1}(i-n+1)} \\ &= \frac{p_{i+1}(i-n+3)(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c + (1-p_{i+1}(i-n+1))k_{i+2}(i-n+2)c}{p_{i+1}(i-n+3)(q_{i+2}(i-n+4) - q_{i+2}(i-n+2)) + (1-p_{i+1}(i-n+1))q_{i+2}(i-n+2)}. \end{aligned}$$

Now

$$v_{i+1}^*(m+1) = \frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)}$$

and

$$v_{i+1}^*(m+3) = \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4) - q_{i+2}(i-n+2)}.$$

By the induction hypothesis, $v_{i+1}^*(m+1) \geq v_{i+1}^*(m+3)$ and thus

$$\frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)} \geq \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4) - q_{i+2}(i-n+2)}.$$

Combining this with Lemma 1 shows that

$$\frac{p_{i+1}(i-n+3)(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c + (1-p_{i+1}(i-n+1))k_{i+2}(i-n+2)c}{p_{i+1}(i-n+3)(q_{i+2}(i-n+4) - q_{i+2}(i-n+2)) + (1-p_{i+1}(i-n+1))q_{i+2}(i-n+2)} \\ \leq \frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)}.$$

From this it follows that

$$v_i^*(m+2) \leq \frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)} \leq v_i^*(m).$$

*Case 2*: Suppose $m = i-n+2$. Then

$$v_i^*(m) = \frac{p_{i+1}(i-n+3)(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c + (1-p_{i+1}(i-n+1))k_{i+2}(i-n+2)c}{p_{i+1}(i-n+3)(q_{i+2}(i-n+4) - q_{i+2}(i-n+2)) + (1-p_{i+1}(i-n+1))q_{i+2}(i-n+2)}.$$

Since we have seen that

$$\frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)} \geq \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4) - q_{i+2}(i-n+2)},$$

it follows from Lemma 1 that

$$v_i^*(m) \geq \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{(q_{i+2}(i-n+4) - q_{i+2}(i-n+2))}.$$

Also note that

$$\begin{aligned} v_i^*(m+2) &= \frac{(k_{i+1}(i-n+5) - k_{i+1}(i-n+3))c}{q_{i+1}(i-n+5) - q_{i+1}(i-n+3)} \\ &= \frac{p_{i+1}(i-n+5)(k_{i+2}(i-n+6) - k_{i+2}(i-n+4))c + (1-p_{i+1}(i-n+3))(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{p_{i+1}(i-n+5)(q_{i+2}(i-n+6) - q_{i+2}(i-n+4)) + (1-p_{i+1}(i-n+3))(q_{i+2}(i-n+4) - q_{i+2}(i-n+2))}. \end{aligned}$$

Now

$$v_{i+1}^*(m+1) = \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4) - q_{i+2}(i-n+2)}$$

and

$$v_{i+1}^*(m+3) = \frac{(k_{i+2}(i-n+6) - k_{i+2}(i-n+4))c}{q_{i+2}(i-n+6) - q_{i+2}(i-n+4)}.$$

By the induction hypothesis, $v_{i+1}^*(m+1) \geq v_{i+1}^*(m+3)$ and thus

$$\frac{(k_{i+2}(i-n+4)-k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4)-q_{i+2}(i-n+2)} \geq \frac{(k_{i+2}(i-n+6)-k_{i+2}(i-n+4))c}{q_{i+2}(i-n+6)-q_{i+2}(i-n+4)}.$$

Combining this with Lemma 1 shows that

$$\begin{aligned}&\frac{p_{i+1}(i-n+5)(k_{i+2}(i-n+6)-k_{i+2}(i-n+4))c+(1-p_{i+1}(i-n+3))(k_{i+2}(i-n+4)-k_{i+2}(i-n+2))c}{p_{i+1}(i-n+5)(q_{i+2}(i-n+6)-q_{i+2}(i-n+4))+(1-p_{i+1}(i-n+3))(q_{i+2}(i-n+4)-q_{i+2}(i-n+2))}\\ &\quad\leq \frac{(k_{i+2}(i-n+4)-k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4)-q_{i+2}(i-n+2)}.\end{aligned}$$

From this it follows that

$$v_i^*(m+2) \leq \frac{(k_{i+2}(i-n+4)-k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4)-q_{i+2}(i-n+2)} \leq v_i^*(m).$$

*Case 3*: Suppose $m > i-n+2$. Then

$$\begin{aligned}v_i^*(m) &= \frac{(k_{i+1}(m+1)-k_{i+1}(m-1))c}{q_{i+1}(m+1)-q_{i+1}(m-1)}\\ &= \frac{p_{i+1}(m+1)(k_{i+2}(m+2)-k_{i+2}(m))c+(1-p_{i+1}(m-1))(k_{i+2}(m)-k_{i+2}(m-2))c}{p_{i+1}(m+1)(q_{i+2}(m+2)-q_{i+2}(m))+(1-p_{i+1}(m-1))(q_{i+2}(m)-q_{i+2}(m-2))}\end{aligned}$$

and

$$\begin{aligned}&v_i^*(m+2)\\ &\quad= \frac{(k_{i+1}(m+3)-k_{i+1}(m+1))c}{q_{i+1}(m+3)-q_{i+1}(m+1)}\\ &\quad= \frac{p_{i+1}(m+3)(k_{i+2}(m+4)-k_{i+2}(m+2))c+(1-p_{i+1}(m+1))(k_{i+2}(m+2)-k_{i+2}(m))c}{p_{i+1}(m+3)(q_{i+2}(m+4)-q_{i+2}(m+2))+(1-p_{i+1}(m+1))(q_{i+2}(m+2)-q_{i+2}(m))}.\end{aligned}$$

Now

$$v_{i+1}^*(m-1) = \frac{(k_{i+2}(m)-k_{i+2}(m-2))c}{q_{i+2}(m)-q_{i+2}(m-2)},$$

$$v_{i+1}^*(m+1) = \frac{(k_{i+2}(m+2)-k_{i+2}(m))c}{q_{i+2}(m+2)-q_{i+2}(m)},$$

and

$$v_{i+1}^*(m+3) = \frac{(k_{i+2}(m+4)-k_{i+2}(m+2))c}{q_{i+2}(m+4)-q_{i+2}(m+2)}.$$

By the induction hypothesis, $v_{i+1}^*(m-1) \geq v_{i+1}^*(m+1) \geq v_{i+1}^*(m+3)$. Thus

$$\frac{(k_{i+2}(m) - k_{i+2}(m-2))c}{q_{i+2}(m) - q_{i+2}(m-2)} \geq \frac{(k_{i+2}(m+2) - k_{i+2}(m))c}{q_{i+2}(m+2) - q_{i+2}(m)} \geq \frac{(k_{i+2}(m+4) - k_{i+2}(m+2))c}{q_{i+2}(m+4) - q_{i+2}(m+2)}.$$

Combining this with Lemma 1 shows that

$$\begin{aligned}&\frac{p_{i+1}(m+1)(k_{i+2}(m+2) - k_{i+2}(m))c + (1 - p_{i+1}(m-1))(k_{i+2}(m) - k_{i+2}(m-2))c}{p_{i+1}(m+1)(q_{i+2}(m+2) - q_{i+2}(m)) + (1 - p_{i+1}(m-1))(q_{i+2}(m) - q_{i+2}(m-2))}\\ &\quad\geq \frac{(k_{i+2}(m+2) - k_{i+2}(m))c}{q_{i+2}(m+2) - q_{i+2}(m)}\\ &\quad\geq \frac{p_{i+1}(m+3)(k_{i+2}(m+4) - k_{i+2}(m+2))c + (1 - p_{i+1}(m+1))(k_{i+2}(m+2) - k_{i+2}(m))c}{p_{i+1}(m+3)(q_{i+2}(m+4) - q_{i+2}(m+2)) + (1 - p_{i+1}(m+1))(q_{i+2}(m+2) - q_{i+2}(m))}.\end{aligned}$$

From this it follows that

$$v_i^*(m+2) \leq \frac{(k_{i+1}(m+2) - k_{i+1}(m))c}{q_{i+1}(m+2) - q_{i+1}(m)} \leq v_i^*(m).$$

Thus $v_i^*(m) \geq v_i^*(m+2)$ holds for any tally $m$ such that $m+2 < 0$ and player $i$ may be called on to vote at the tallies $m$ and $m+2$.

Now consider a tally $m$ such that $m+2 \geq 0$ and player $i$ may be called on to vote at the tallies $m$ and $m+2$. Note that if $m \leq 0$, then we know from Proposition 3 that $v_i^*(m) \geq 0 \geq v_i^*(m+2)$. Also, if $m > 0$, then $-m < 0$, $-(m+2) < 0$, and we know from the previous paragraphs that $v_i^*(-(m+2)) \geq v_i^*(-m)$. Then the fact that $v_i^*(-m) = -v_i^*(m)$ for all $m$ implies that $v_i^*(m) \geq v_i^*(m+2)$. Thus $v_i^*(m) \geq v_i^*(m+2)$ always holds, and $v_i^*(m)$ is non-increasing in $m$. ■

*Proof of Proposition 5*. First note that the results hold if $i = n-1$. Player $i = n-1$ may only ever be called on to vote at tallies of 1 or $-1$ and player $i = n$ may only ever be called on to vote at a tally of 0. Now we know from Proposition 3 that $v_{n-1}^*(1) \leq 0 = v_n^*(0) = 0 \leq v_{n-1}^*(-1)$. Thus the results hold for $i = n-1$.

Now assume that $i > n-1$. I first show the results hold if $m < 0$. There are several cases.

*Case 1*: Suppose $m = i - n$. Then

$$\begin{aligned}v_i^*(m) &= \frac{k_{i+1}(i-n+1)c}{q_{i+1}(i-n+1)} = \frac{(1 + p_{i+1}(i-n+1)k_{i+2}(i-n+2))c}{p_{i+1}(i-n+1)q_{i+2}(i-n+2)}\\ &\geq \frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)} = v_{i+1}^*(m+1).\end{aligned}$$

Also, player $i+1$ can never be called on to vote at a tally of $m-1$, so we do not need to worry about showing that $v_i^*(m) \leq v_{i+1}^*(m-1)$. Thus the results hold if $m = i - n$.

*Case 2*: Suppose $m = i - n + 2$. Then

$$v_i^*(m) = \frac{p_{i+1}(i-n+3)(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c + (1 - p_{i+1}(i-n+1))k_{i+2}(i-n+2)c}{p_{i+1}(i-n+3)(q_{i+2}(i-n+4) - q_{i+2}(i-n+2)) + (1 - p_{i+1}(i-n+1))q_{i+2}(i-n+2)}.$$

Since

$$\frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)} \geq \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4) - q_{i+2}(i-n+2)},$$

it follows from Lemma 1 that

$$\begin{aligned}\frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)} &\\ &\geq \frac{p_{i+1}(i-n+3)(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c + (1-p_{i+1}(i-n+1))k_{i+2}(i-n+2)c}{p_{i+1}(i-n+3)(q_{i+2}(i-n+4) - q_{i+2}(i-n+2)) + (1-p_{i+1}(i-n+1))q_{i+2}(i-n+2)}\\ &\geq \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4) - q_{i+2}(i-n+2)}.\end{aligned}$$

But

$$v_{i+1}^*(m-1) = \frac{k_{i+2}(i-n+2)c}{q_{i+2}(i-n+2)}$$

and

$$v_{i+1}^*(m+1) = \frac{(k_{i+2}(i-n+4) - k_{i+2}(i-n+2))c}{q_{i+2}(i-n+4) - q_{i+2}(i-n+2)}.$$

Thus $v_{i+1}^*(m-1) \geq v_i^*(m) \geq v_{i+1}^*(m+1)$ holds if $m = i - n + 2$.

*Case 3*: Suppose $m > i - n + 2$. Then

$$v_i^*(m) = \frac{p_{i+1}(m+1)(k_{i+2}(m+2) - k_{i+2}(m))c + (1 - p_{i+1}(m-1))(k_{i+2}(m) - k_{i+2}(m-2))c}{p_{i+1}(m+1)(q_{i+2}(m+2) - q_{i+2}(m)) + (1 - p_{i+1}(m-1))(q_{i+2}(m) - q_{i+2}(m-2))}.$$

Since

$$\frac{(k_{i+2}(m) - k_{i+2}(m-2))c}{q_{i+2}(m) - q_{i+2}(m-2)} \geq \frac{(k_{i+2}(m+2) - k_{i+2}(m))c}{q_{i+2}(m+2) - q_{i+2}(m)},$$

we have

$$\begin{aligned}\frac{(k_{i+2}(m) - k_{i+2}(m-2))c}{q_{i+2}(m) - q_{i+2}(m-2)} &\\ &\geq \frac{p_{i+1}(m+1)(k_{i+2}(m+2) - k_{i+2}(m))c + (1 - p_{i+1}(m-1))(k_{i+2}(m) - k_{i+2}(m-2))c}{p_{i+1}(m+1)(q_{i+2}(m+2) - q_{i+2}(m)) + (1 - p_{i+1}(m-1))(q_{i+2}(m) - q_{i+2}(m-2))}\\ &\geq \frac{(k_{i+2}(m+2) - k_{i+2}(m))c}{q_{i+2}(m+2) - q_{i+2}(m)}.\end{aligned}$$

But

$$v_{i+1}^*(m-1) = \frac{(k_{i+2}(m) - k_{i+2}(m-2))c}{q_{i+2}(m) - q_{i+2}(m-2)}$$

and

$$v_{i+1}^*(m+1) = \frac{(k_{i+2}(m+2) - k_{i+2}(m))c}{q_{i+2}(m+2) - q_{i+2}(m)}.$$

Thus $v_{i+1}^*(m-1) \geq v_i^*(m) \geq v_{i+1}^*(m+1)$ holds if $m > i - n + 2$. From this it follows that $v_{i+1}^*(m-1) \geq v_i^*(m) \geq v_{i+1}^*(m+1)$ always holds for $m < 0$.

To see the results hold if $m = 0$, note from Proposition 3 that $v_{i+1}^*(-1) \geq 0 = v_i^*(0) = 0 \geq v_{i+1}^*(1)$. Thus $v_{i+1}^*(m-1) \geq v_i^*(m) \geq v_{i+1}^*(m+1)$ holds if $m = 0$.

Finally I show the results hold if $m > 0$. To see this, note that if $m > 0$, then $-m < 0$, and we know from the results on the case where $m < 0$ that $v_{i+1}^*(-m-1) \geq v_i^*(-m) \geq v_{i+1}^*(-m+1)$. But we know from Proposition 3 that $v_i^*(-m) = -v_i^*(m)$ for all $m$. Thus $v_{i+1}^*(-m-1) \geq v_i^*(-m) \geq v_{i+1}^*(-m+1)$ implies that $-v_{i+1}^*(m+1) \geq -v_i^*(m) \geq -v_{i+1}^*(m-1)$ and $v_{i+1}^*(m-1) \geq v_i^*(m) \geq v_{i+1}^*(m+1)$. So the results hold if $m > 0$. Thus $v_{i+1}^*(m-1) \geq v_i^*(m) \geq v_{i+1}^*(m+1)$ always holds whenever these values are well defined. ■

*Proof of Proposition 6*. To prove this, it suffices to show that if player $i$ is called on to vote at an arbitrary tally $m$, then the probability $A$ wins the election if $v_i = v$ and $v_{i+1} = v'$ is at least as high as the probability $A$ wins the election if $v_i = v'$ and $v_{i+1} = v$. To do this, it suffices to show that $A$ will get at least as many votes from players $i$ and $i + 1$ if $v_i = v$ and $v_{i+1} = v'$ as $A$ will get if $v_i = v'$ and $v_{i+1} = v$. There are five cases.

*Case 1*: Suppose that if $v_i = v$, then player $i$ votes for $B$ and the election finishes with $B$ being elected immediately after $i$'s vote. Then since $v' < v \leq v_i^*(m)$, player $i$ also votes for $B$ if $v_i = v'$, and $B$ is also elected if $v_i = v'$. Thus $A$ does not do any better if $v_i = v'$ than if $v_i = v$.

*Case 2*: Suppose that if $v_i = v$, then player $i$ votes for $A$ and the election finishes with $A$'s election immediately after $i$'s vote. Then $A$ cannot do any better than this, and $A$ will not do better if $v_i = v'$ than if $v_i = v$.

*Case 3*: Suppose that if $v_i = v$, then player $i$ votes for $B$, and player $i + 1$ is called on to vote at a tally of $m - 1$. Then since $v' < v \leq v_i^*(m)$, player $i$ also votes for $B$ if $v_i = v'$, and player $i+1$ is also called on to vote at a tally of $m-1$ if $v_i = v'$. Now since $v \leq v_i^*(m) \leq v_{i+1}^*(m-1)$, player $i + 1$ votes for $B$ at a tally of $m - 1$ if $v_{i+1} = v$. Thus if $v_i = v'$ and $v_{i+1} = v$, then both players $i$ and $i + 1$ vote for $B$. But this means that $A$ cannot do any worse among players $i$ and $i + 1$ than $A$ does when $v_i = v'$ and $v_{i+1} = v$, so the probability $A$ wins the election if $v_i = v$ and $v_{i+1} = v'$ is at least as high as the probability $A$ wins the election if $v_i = v'$ and $v_{i+1} = v$.

*Case 4*: Suppose that if $v_i = v$ and $v_{i+1} = v'$, then player $i$ votes for $A$, player $i + 1$ is called on to vote at a tally of $m + 1$, and player $i + 1$ votes for $A$. Then $A$ cannot do any better among voters $i$ and $i + 1$ than $A$ does when $v_i = v$ and $v_{i+1} = v'$, so the probability $A$ wins the election if $v_i = v$ and $v_{i+1} = v'$ is at least as high as the probability $A$ wins the election if $v_i = v'$ and $v_{i+1} = v$.

*Case 5*: Suppose that if $v_i = v$ and $v_{i+1} = v'$, then player $i$ votes for $A$, player $i+1$ is called on to vote at a tally of $m+1$, and player $i+1$ votes for $B$. Then $v' \leq v^*_{i+1}(m+1) \leq v^*_i(m)$. So if $v_i = v'$, then player $i$ votes for $B$. Thus if $v_i = v'$ and $v_{i+1} = v$, then $B$ gets at least as many votes from players $i$ and $i+1$ as $B$ gets if $v_i = v$ and $v_{i+1} = v'$. Thus the probability $A$ wins the election if $v_i = v$ and $v_{i+1} = v'$ is at least as high as the probability $A$ wins the election if $v_i = v'$ and $v_{i+1} = v$. From this it follows that the probability $A$ wins the election if $v_i = v$ and $v_{i+1} = v'$ is always at least as high as the probability $A$ wins the election if $v_i = v'$ and $v_{i+1} = v$. ■

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## REFERENCES

Aldrich, J. H., 1980, A dynamic model of presidential nomination campaigns. *American Political Science Review* 74, 651–669.

Ali, S. N. and N. Kartik, 2008, Social learning and elections. Typescript, University of California, San Diego.

Bartels, L. M., 1985, Expectations and preferences in presidential nominating campaigns. *American Political Science Review* 79, 804–815.

———, 1988, *Presidential Primaries and the Dynamics of Public Choice* (Princeton University Press, Princeton, NJ).

Battaglini, M., 2005, Sequential voting with abstention. *Games and Economic Behavior* 51, 445–463.

———, R. Morton, and T. Palfrey, 2007, Efficiency, equity, and timing of voting mechanisms. *American Political Science Review* 101, 409–424.

Buell, E. H., 1986, Divisive primaries and participation in fall presidential campaigns: a study of 1984 New Hampshire primary activists. *American Politics Research* 14, 376–390.

Callander, S., 2007, Bandwagons and momentum in sequential voting. *Review of Economics Studies* 74, 653–684.

Carter, J., 1982, *Keeping Faith: Memoirs of a President* (Bantam, New York).

Dekel, E. and M. Piccione, 2000, Sequential voting procedures in symmetric binary elections. *Journal of Political Economy* 108, 34–55.

Hummel, P., 2009, Sequential voting in large elections with multiple candidates. Typescript, Stanford University.

Kenney, P. J. and T. W. Rice, 1987, The relationship between divisive primaries and general election outcomes. *American Journal of Political Science* 31, 31–44.

——— and ———, 1994, The psychology of political momentum. *Political Research Quarterly* 47, 923–938.

Klumpp, T. and M. K. Polborn, 2006, Primaries and the New Hampshire effect. *Journal of Public Economics* 90, 1073–1114.

Knight, B. and N. Schiff, 2007, Momentum in presidential primaries. Typescript, Brown University.

Lengle, J. I., 1980, Divisive presidential primaries and party electoral prospects, 1932–1976. *American Political Research* 8, 261–277.

———, D. Owen, and M. W. Sonner, 1995, Divisive nominating mechanisms and Democratic party electoral prospects. *Journal of Politics* 57, 370–383.

Miller, P. M., M. E. Jewell, and L. Sigelman, 1988, Divisive primaries and party activists: Kentucky, 1979 and 1983. *Journal of Politics* 50, 459–470.

Morton, R. B. and K. C. Williams, 1999, Information asymmetries and simultaneous versus sequential voting. *American Political Science Review* 93, 51–67.

——— and ———, 2001, *Learning by Voting: Sequential Choices in Presidential Primaries and Other Elections* (University of Michigan Press, Ann Arbor, MI).

Popkin, S. L., 1994, *The Reasoning Voter: Communication and Persuasion in Presidential Campaigns* (University of Chicago Press, Chicago, IL).

Southwell, P. L., 1986, The politics of disgruntlement: nonvoting and defection among supporters of nomination losers, 1968–1984. *Political Behavior* 8, 81–95.

Stone, W. J., 1984, Prenomination candidate choice and general election behavior: Iowa presidential activists in 1980. *American Journal of Political Science* 28, 361–378.

———, 1986, The carryover effect in presidential elections. *American Political Science Review* 80, 271–279.

———, L. R. Atkeson, and R. B. Rapoport, 1992, Turning on or turning off? Mobilization and demobilization effects of participation in presidential nomination campaigns. *American Journal of Political Science* 36, 665–691.

Strumpf, K. S., 2002, Strategic competition in sequential election contests. *Public Choice* 111, 377–397.